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Video Case Analysis of Students' Mathematical Thinking: Initial Development Process

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Abstract: Video shows promise for supporting mathematics teacher candidates' understanding of student thinking. As a result, we began a design-based research project on using video-based online learning modules as a component of a teacher preparation program to improve candidates' ability to recognize and make connections between patterns in students' informal and formal reasoning. In this paper, we describe the initial development process of the video case analysis of student thinking (VCAST) modules and then share the results associated with developing our first two modules on functional reasoning. This design-based research project is informed by literature on student thinking and involves a purposeful selection of authentic examples of student reasoning. The results indicate that our process led to an adequate range of student ideas from which to build the modules. Moving forward, the research team is continuing this work by investigating additional cycles of development, improvement, and implementation of these online modules.

Introduction

Instruction which progressively builds upon students' ideas increases the accessibility of mathematical ideas to a broader range of students and results in a more equitable practice (Boaler, 2004). Further, when teachers build upon students' thinking to develop their understanding of mathematics (e.g. Carpenter, Franke, Jacobs, Fennema, & Empson, 1998), students' conceptual understanding of mathematics is improved while procedural fluency is maintained (e.g., Fennema et al., 1996). While there are advantages to developing this type of knowledge in teacher candidates through field-based work, there are limitations associated with doing so through field-based work alone (Ko, Milewski., & Herbst, 2017). It was these limitations that motivated us to begin developing a series of video-based online learning modules to improve secondary mathematics teacher candidates' ability to attend to students' reasoning. The work we report on here is part of a 4-year design-based research project aimed at iteratively improving upon the design and use of a series of video-based online modules focused on functions and modeling ideas (Lesh, Kelly, & Yoon, 2008). In this chapter, we describe the first two phases of the design-based research approach we used to develop two video-based online modules that feature video case analysis of student thinking (VCAST).

The VCAST project aims to use authentic cases of secondary student functional reasoning to support candidates' knowledge of the range of informal ideas students have that form the basis for building more formal mathematical ideas. Because of this, our module development process involves conducting one-on-one interviews with secondary students as they complete mathematical tasks and then curating the resulting videos and student artifacts for module content. The guiding research questions that drove our initial development process are:

1. What does the literature reveal about student reasoning in functions?
2. To what extent does the evidence of students' functional reasoning represent a range of student reasoning?

Theoretical Perspective

In this section, we briefly describe the four areas of work closely aligned to our project: (1) specialized knowledge of mathematics for teaching, (2) functional reasoning, (3) video and teacher education, and (4) design-based research.

Specialized Knowledge of Mathematics for Teaching

To advance student understanding, teachers must pay careful attention to students' mathematical ideas and respond to students in ways that enable them to build upon their ideas (Ball, Thames, & Phelps, 2008). Learning to attend and respond in this way requires a specialized knowledge of how mathematical ideas are related, how to represent ideas in meaningful ways, and common patterns in students' reasoning (Ball, Thames, & Phelps, 2008). Without this type of knowledge, teachers' attention to students' ideas can result in teachers doing the work for students (if the student erred) or congratulating the student (if the student used a correct strategy), rather than pressing the student to reformulate or extend their ideas (Henningsen, & Stein, 1997).

Functional Reasoning

A function in mathematics is defined as "a correspondence between two nonempty sets that assigns to every element of the first set (the domain) exactly one element in the second set (the codomain)" (Vinner & Dreyfus, 1989, p. 357). Function is a unifying concept in K-12 mathematics that applies to the study of algebra, geometry, probability, and statistics. However, many secondary students leave high school with impoverished reasoning abilities with functions, equating functions with a single rule or equation and experiencing difficulty in generalizing functional relationships between quantities (Carlson, 1998; Thompson, 1994).

Understanding a concept in mathematics is achieved when one is able to use, identify, apply, generalize and create extensions of that concept (Sierpinska, 1992). For the concept of function, understanding is evidenced by the ability to work with functions in a variety of ways, including:

(1) using function notation,
(2) recognizing a function as a dynamic process that maps each element from a set of inputs to a single element in the set of outputs,
(3) building and working with functions as mathematical models of relationships between quantities,
(4) interpreting and making connections between multiple representations of functions,
(5) coordinating changes in quantities that covary,
(6) manipulating functions as abstract objects, and
(7) identifying and using structural generalizations for families of functions.

The modules we are developing are designed to address items (1) through (5).

Video and Teacher Education

Over the years, teacher education programs have placed a greater emphasis on field experiences for prospective teachers (Zeichner, 1981). One of the benefits is the opportunity for prospective teachers to see what teaching and learning look like in real classrooms (Wilson, Floden, & Ferrini-Mundy, 2001). As beneficial as field experiences can be, the variance in their quality makes it difficult to concentrate on specific "teachable moments." As a result, teacher educators have explored different ways to use video in teacher education to help slow down and capture these teachable moments (e.g., Sherin & van Es, 2005). In addition, videos of teachable moments can help prospective teachers not only "see" important events but also pause, reflect, and even rewatch them (Coffey, 2014). Our use of video to prepare teachers has been influenced by the work of multiple researchers (e.g., Philipp et al., 2007; Santagata, 2014; Star & Strickland, 2008).

Design-based Research

Informed by other uses of video (Walkoe, 2015; Sherin & van Es, 2005) and research on technology-enhanced learning environments (c.f. Wang & Hannafin, 2005) we recognize that technology is not a panacea; it is the pedagogy and instructional design that make a difference in learning outcomes (Blomberg, Sherin, Renkl, Glogger-Frey, & Seidel, 2014). Following Reeves (2006), we have set out to design, develop, and improve the use of our modules—and in turn candidate learning—through a design-based research approach that involves (1) analysis of the problem, (2) module design, (3) iterative cycles of module testing and refinement, and (4) reflection to produce design principles. Design research methodology (Collins, Joseph, & Bielaczye, 2004; Reeves, Herrington, & Oliver, 2005) is particularly applicable for curriculum development as it provides "an avenue for studying learning within the complexity of interacting educational systems" (Zawojewski, Chamberlin, Hjalmarson, & Lewis, 2008, p. 220) and has been effectively used in mathematics education to investigate student learning (c.f. Lobato, 2008) and teachers' development of practice (c.f. Zawojewski et al., 2008).

Video Case Analysis of Student Thinking (VCAST) Modules

Each online module incorporates short video clips of secondary students working on a mathematical task and a series of questions focused on the students' mathematical reasoning while solving the task. The overarching learning goal with each module is for candidates to recognize and make connections between secondary students' informal and formal reasoning about key ideas of functions and modeling. The two completed modules discussed in this chapter feature students working with figural patterns (Module I) and graphs of discrete data (Module II).

The modules are designed to be used in various formats. While we are specifically interested in using them in face-to-face or hybrid undergraduate courses with the intent that candidates complete each module before engaging in a face-to-face follow-up session, they could also be used in fully online courses. Currently, the modules are embedded in an upper-division mathematics course. The general structure of each module engages candidates in a cycle of different design elements. First, candidates solve the mathematical task and provide a written explanation of their work. Next, candidates either examine written student work or watch short video clips of students working on the task. Then, based on the evidence provided, candidates are asked to formulate hypotheses about the secondary students' reasoning. Once these hypotheses are made, additional student evidence is provided, either written work or subsequent video clips. Candidates then have the opportunity to revise their initial hypotheses about the students' reasoning. Finally, candidates are prompted to reflect on what they have learned.

After completion of the module, and prior to the follow-up session, candidates read a research-based article associated with tasks related to the one featured in the module. During follow-up sessions, candidates work in small groups on identifying/describing student approaches, making inferences about student understanding, and relating the elicited student ideas to features of the module task. The follow-up sessions also involve opportunities to examine additional student work, usually in written form so that by the conclusion of the follow-up session, candidates are in a position to engage in a whole group discussion about connections between the assigned reading and a broad collection of examples of student reasoning.

Module I: The Hexagon Task

Figural pattern tasks such as the hexagon task in Figure 1 can be used to elicit students' ideas about building functions that model a situation. Figural patterns allow students to make algebraic generalizations in a variety of ways, potentially leading to the meaningful use of variables and expressions with the quantities involved. In particular, near generalization tasks can be solved by step-by-step drawing or counting (e.g., "What is the perimeter of the 5^{th} figure in the hexagon task below?"), whereas far generalization tasks require determining a pattern that can be quickly applied to a larger number, such as the hexagon task in Figure 1 (Stacey, 1989).

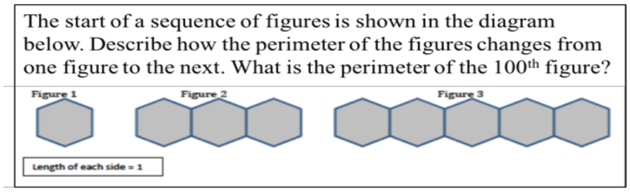

Figure 1. The Hexagon Task (Adapted from Hendrickson, Honey, Juehl, Lemon, & Sutorios, 2012)

Module II: The Bus Stop Task

Graphing tasks such as the bus stop task in Figure 2 can be used to elicit evidence of students' covariational reasoning. As noted in various research studies focused on the development of functional reasoning (Confrey & Smith, 1995; Oehrtman, Carlson & Thompson, 2008; Saldanha & Thompson, 1998), students' ability to coordinate the relationship between successive values in two sequences and then to couple two quantities into a singular object that can be represented graphically is both developmental and nontrivial. In this task, students are asked to coordinate the static quantities of height and age for a group of seven individuals represented pictorially and then to represent this coordination in a Cartesian graph.

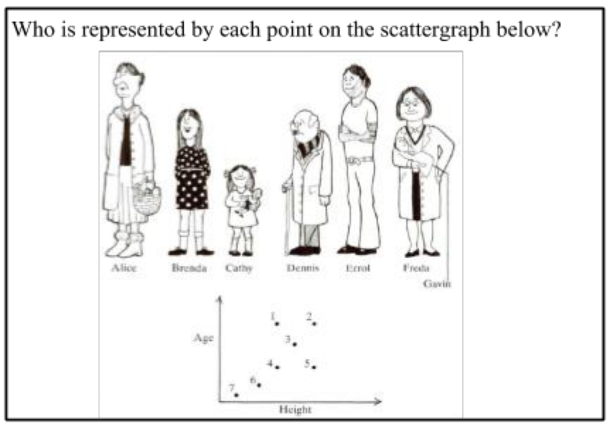

Figure 2. The Bus Stop Task (Taken directly from the book, *The Language of Functions and Graphs*, published by the Shell Centre for Mathematical Education, 1988)

Methods for Initial Development

The initial development process for the VCAST modules described here involved two phases (See Table 1). In the first phase, we consulted the research literature on student reasoning for a key idea related to functions and modeling (i.e., Module I: figural patterns; Module II: covariational reasoning), and then selected a task that could be used with secondary students to elicit a range of thinking for the identified key idea. An important driving question in this part of the first phase is, "What does research evidence reveal about how student thinking progresses?" Researchers next conducted one-on-one interviews with secondary students, analyzed the student work (evidenced by video & student artifacts) in relation to the literature on student thinking, and developed codes relevant to a hypothesized progression of learning. Following Reeves (2006), our intent in this first phase was to clarify the learning problem for teacher candidates. In our case, that meant clarifying what is known about student reasoning for a particular key idea in mathematics (e.g. figural pattern tasks) and to what extent we were able to capture samples of students' functional reasoning along a broad progression of learning. The problem was then articulated around how best to engage teacher candidates in developing understanding of the student thinking captured.

In the second phase, three researchers selected a range of representative samples to potentially use for video cases, independently developed a proposed module structure, and then worked together to blend selected elements from the proposed module structures. Module learning goals were drafted at the beginning of the process and then revised.

Phase 1: Analysis of the Problem	**Phase 2: Module Design**
Review related literature on students functional reasoning and related teacher learning	Create a table to display the codes for reasoning elicited by each student
Select a functional reasoning task that has potential to elicit a range of student thinking	Select potential video cases that represent the full range of student reasoning
Capture student thinking during one-on-one interviews and analyze data in relation to the literature	Independently develop three possible module structures for consideration.
Develop potential progression of student learning and identify opportunities for candidate learning	Blend desired elements from drafted module structures and revise learning goals for candidates

Table 1. Initial Development Process Phases

We then interviewed 30 participants: 14 students from a midsized, urban high school and 16 students from a midsized, rural middle school. Both schools were located in the western United States. Students ranged in age from 12 to 18 years and all were familiar with the mathematical topics addressed during the interviews. Data collection occurred at the schools and consisted of video-recorded, individual student interviews as they worked on three to four mathematical tasks. Students spent about five to ten minutes of the 30-minute interview working on either the bus stop task (middle school) or the hexagon task (high school). As students worked through each task, a researcher asked students to document their work on paper and to explain their thinking aloud. The researcher also asked questions to support student progress.

Qualitative analysis of student work involved creating a set of codes for categories of reasoning that were independently applied by at least two researchers. Discrepancies in coding were negotiated among all of the researchers until agreement was reached. Depending on the task, the list of codes for a task was either clearly informed by the literature on student thinking (e.g. Rivera, 2010) or developed through a grounded theory approach (Strauss & Corbin, 1994) of creating and revising codes based on categories of reasoning anticipated by the researchers.

Results and Discussion

The Hexagon Task Results

Approaches to figural pattern tasks can be described in relation to four interrelated categories that depend on the representations students choose for the elements in the sequence (Sequence Elements) and their symbolic generalizations (Symbolic Generalization) (See Figure 3 and Figure 4). With respect to the hexagon task, students may approach the problem by generating a numerical sequence that corresponds to the perimeter of the figure for the first several figures and then base their reasoning on patterns they notice in the numerical sequence. When doing so, they are using a numerical representation of the sequence elements. They may then generalize patterns either by stating a recursive pattern that describes how the values change from one element to the next (Numerical & Recursive), or a functional pattern that describes how to generate a sequence element based on the sequence position (Numerical & Functional). When students observe patterns related to the geometric features of the figures, they are using a figural representation of the sequence elements. These patterns may focus on how the hexagon figures change from one figure to the next (Figural & Recursive), or how the perimeter can be determined based on either the number of hexagons or the figure number (Figural & Functional).

The high school students interviewed used a variety of approaches when solving the hexagon task. Figure 4 illustrates the range of approaches students demonstrated. When solving the hexagon task, some students reasoned through the task using one of the four approaches (Students A, C, F, K) or some combination (Students B, D, E, G, H, I, J, L). For example, one student noticed the number of hexagons increases by two for each figure and then used that information to determine an explicit functional relationship for the perimeter in terms of the number of hexagons (Student B). Other students started by observing the increase in perimeter for each figure and then used that information to generate an explicit function for the perimeter of each figure (Students H, I, J, L). And other students started by trying to write a function for the perimeter based on the number of hexagons (observing how the hexagons in each figure contribute to the perimeter) and then used a numerical approach to determine the number of hexagons in each figure (Students D, E, G). What is important is that there are identifiable patterns in how the students reasoned about the task that can be built upon in instruction.

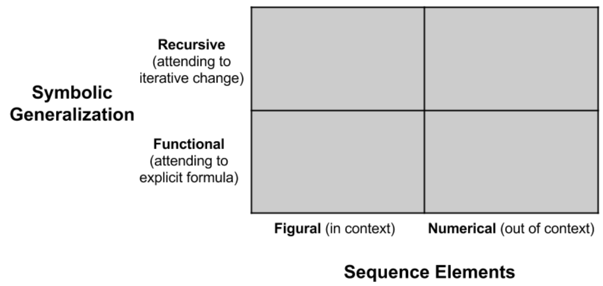

Figure 3. Figural Pattern Approaches

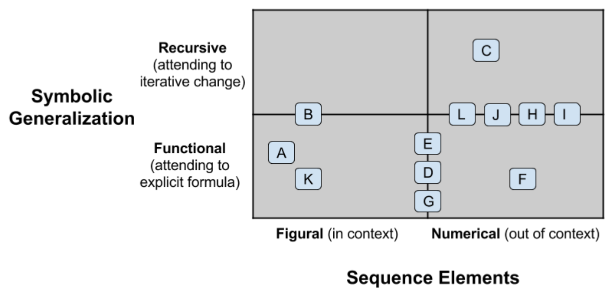

Figure 4. High School Student Approaches

As we conducted our analyses, we developed a list of potential opportunities for candidate learning: (1) deciphering students' cryptic notations; (2) recognizing what is correct in a slightly flawed strategy, (3) recognizing ideas related to an efficient strategy, and (4) making connections between different strategies. We also wanted the module to introduce candidates to a range of ways to approach the hexagon task, but not necessarily all possibilities. We suspected that candidates would be familiar with a numerical approach and less familiar with figural approaches.

The list of potential opportunities for learning, together with the analyses of students' approaches, led to the articulation of learning goals for the module. Specifically, Module I is designed to support candidates' ability to:

- Listen and attend to students' mathematics as they make generalizations about a sequence of geometric figures
- Articulate students' mathematics about figural and numerical generalizations for a given sequence and how those generalizations are reflected in students' work
- Recognize correct reasoning expressed via informal language and vocabulary that describes figural and numerical generalizations
- Describe connections between candidates' mathematical approaches and students' approaches

These learning goals were articulated and revised in the at the end of the second phase of our process for developing module content.

We selected the work of Student A (Figural & Functional) and Student J (Numerical & Recursive and Numerical & Functional) from which to build case studies for Module I. Because we anticipated candidates would be less familiar with Student A's approach, we put Student J's case study before Student A's in the module. Recall, candidates work through the same task at the beginning of the module. Thus, starting with student work that is potentially more familiar should facilitate candidates' progress through the first part of the module. We structured each case study in a similar way, starting with the candidates making a hypothesis about the student's approach to the hexagon task based on either a short video clip of the student sharing their thinking (Student J) or based upon an image of the student's work (Student A). Each case study was designed to then present additional information about the student's reasoning and prompt candidates to revise their hypotheses. The case studies conclude with a prompt for candidates to compare their own approach to the hexagon task to the student's approach.

The Bus Stop Task Results

Student approaches to the bus stop task can be categorized by the evidence they produced related to four key conceptual understandings: (1) ordering quantities without assigned values, (2) utilizing graphical conventions as they relate to the assigned meaning of horizontal locations in the Cartesian plane, (3) utilizing graphical conventions as they relate to the assigned meaning of vertical locations in the plane, and (4) connecting the coordination of paired covarying quantity values to their representations in the plane.

As anticipated, our interviews of middle school students revealed a range of thinking about the bus stop task. Table 2 provides a summary of our analysis of the collected video and written student data. One student was unable to detach the meaning of a point's location on the plane from its numerical label and vacillated between having the label represent height or age. Other students (n = 4) attended solely to height and ordered the people represented in the task accordingly. When they encountered two individuals with the same height, they either used age or their position in the picture to "break the tie." In contrast, those who exhibited nascent covariational reasoning (n = 19) appeared to understand that a single point in the plane carried information about paired quantities. They illustrated this understanding by attending to both height and age as they matched individuals to graphed points. Those who were able to maintain a stable assignment of quantity to graphical coordinate (n = 7) were successful with the task, though 2 of these 7 corrected their thinking following an interviewer's clarifying question about their approach. Others, for whom the quantity to coordinate assignment was less stable (n = 12), were successful with the task until they encountered individuals or points with a shared quantity value. A trend that emerged from our analysis indicates that assigning quantities typically represented vertically (such as height) to a horizontal coordinate in a graphical representation is challenging for middle school students.

Conceptual Understanding	**Brief Description as it Relates to the Bus Stop Task**	**Coding Options and Percentage Receiving Code**
A single point can represent the values of two different quantities.	Student understands that each point in the Cartesian plane supplies both age and height information.	Not demonstrated (n = 4)
		Partially demonstrated (n = 4)
		Clearly demonstrated (n = 15)
One-dimensional graphical conventions apply to all vertical axes or number lines.	Student understands that moving up corresponds to increasing age, while moving down corresponds to decreasing age.	Never (n = 5)
		Sometimes correct (n = 11)
		Always correct (n = 7)
One-dimensional graphical conventions apply to all horizontal axes or number lines.	Student understands that moving right corresponds to increasing height, while moving left corresponds to decreasing height.	Never (n = 4)
		Sometimes correct (n = 12)
		Always correct (n = 7)
Quantities do not require numerical values to be relatively positioned along an unscaled axis.	Student can compare and order relative heights and/or ages that do not have assigned numerical values.	Not demonstrated (n = 1)
		Clearly demonstrated (n = 22)

Table 2. Bus Stop Task Summary of Student Reasoning (n = 23)

While we conducted our analyses, we again developed a list of potential opportunities for candidate learning as it relates to recognizing (1) differences between students' bivariate and univariate reasoning, (2) the evidence of reasoning students' gestures and verbal explanations provide, (3) how features of tasks can elicit student evidence with potential to be misinterpreted, and (4) how assignment of height quantities to horizontal coordinates can reveal weaknesses in students' covariational reasoning flexibility. We also wanted the module to reveal the challenges inherent to gauging the difficulty of a task.

This list, combined with our analysis of student approaches, led to the articulation of learning goals for the module. Specifically, in the process of developing the module content, Module II was designed to support candidates' ability to:

- Listen and attend to students' mathematics as they reason about bivariate data associated with a static situation involving unmeasured quantities.
- Articulate students' mathematical reasoning about the relative location of points in two dimensions within a context.
- Recognize correct reasoning expressed via gestures and informal language that describes how the relative location of points in two dimensions can be used to draw contextual conclusions.
- Recognize how features of a task can influence its difficulty and impact the reasoning exhibited by students.

We selected the work of three different middle school students for Module II's case studies. Because we anticipated that candidates would struggle to accurately gauge the task's difficulty, the module asks them to identify components of the task they deem easy and challenging and to rank the difficulty on a scale from 1 to 10. We also anticipated that students' initial assignment of points 7, 6, and 5 might lead candidates to make faulty assumptions about students' understanding. For this reason, we provided three video clips which feature students starting the task in different ways and then ask the candidates to predict who will be the most successful. We then provide the subsequent video clips for these same three students so candidates can revise their initial hypotheses and also to highlight the ways in which a student can (a) reason correctly and still arrive at an incorrect answer when one of the unmeasured quantities (age) is challenging to estimate, (b) correct errors while progressing through a task, and (c) reason correctly but then falter when cognitive disequilibrium is triggered (vertical measures such as heights can be assigned to horizontal coordinates in the plane). At the end of the module, candidates are again asked to rank the task difficulty and to make a prediction about each student's approach to a revised version of the task.

Conclusion

The goal of this design-based research project is to develop video-based online learning modules to support secondary mathematics teacher candidates' ability to attend to students' mathematical ideas. As with any technological tool for teaching, what matters most is how learners are directed to use the tool (Reeves, 2006). The use of a framework that focuses the intent and selection of video, along with the direction and assessment of teachers' use of video, is a critical component of success (Santagata, 2014; Star & Strickland, 2008). The study described in this chapter, in particular, provides an example of one way to purposely design video-based instructional modules informed by literature on student thinking and built around authentic examples of student reasoning. The process we described in this chapter contributes not only to forming the basis for module content but also to developing a content framework which, in turn, supports our efforts to clarify candidate learning goals and influences module design.

To iteratively improve the VCAST modules, the research team will conduct a series of design experiments using mixed methods to investigate the development, improvement, and implementation of online modules focused on engaging candidates in recognizing and making connections between patterns in students' mathematical reasoning. For instance, the modules are being used in an undergraduate course in the fall of 2018 to further test the modules. The content framework for each module, along with other theoretical underpinnings for this work (c.f. Carney, Cavey, & Hughes, 2017), put us in an ideal position to test and refine module content. In a period where programs are looking for ways to integrate subject matter preparation with learning about teaching practices, it is critical to have empirically-based models for such learning.

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